First Passage Percolation

Christopher Hoffman

Scribe: Avi Levy
Illustrations: Brent Werness

Math 582G, Winter 2016, UW

February 26, 2016

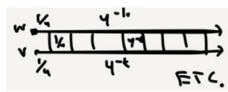
JANUARY 4

Thanks to Jacob Richey for sharing his notes from the first day.

We are given a graph $G = (V, E)$ and a probability distribution D on $(0, \infty)$. Let $\{w(e)\}_{e \in E}$ be iid random variables with distribution D. We construct a random metric space on G using the weights $w(e)$ as follows:

- For $v, w \in E$, a path P from v to w is a sequence of vertices $v = v_0, v_1, \ldots, v_n = w$ with $v_i v_{i+1} \in E$.
- Let $\gamma(P) = \sum_{i=0}^{n-1} w(v_i v_{i+1})$.
- Set $T(v, w) = \inf\{\gamma(P) \colon v \to_P w\}$.

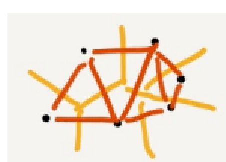
Paths for which the infimum is attained are **geodesics**. Do geodesics exist and are they unique?



For the graph with (deterministic) edge weights given above, the infimum is not attained: there is no geodesic from v to w. Even when geodesics exist, they may not be unique (consider $\mathbb{Z}^d$ with all weights 1). However, when G has uniformly bounded degree, V is countable, and D is continuous, geodesics exist and are a.s. unique.

Typical choices for G and D:

1. $G = \mathbb{Z}^d$ with edges between nearest neighbors for $d \geq 2$, especially the case $d = 2$. (The case $d = 1$ is trivial.)
2. G is a random graph (e.g. $G = G(n, p)$ Erdős-Rényi)
3. G is a Delaney triangulation (Voronoi tesselation of a Poisson point process in $\mathbb{R}^d$), nice because it is spherically symmetric.



4. The hypercube graph $G = \{0, 1\}^n$
5. $D = \exp(1)$
6. Continuous distributions with a moment condition
7. Stationary distributions (in place of iid)

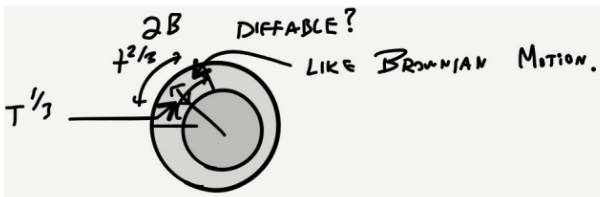
For $G = \mathbb{Z}^d$, consider the ball

$$B(0,t) = \{z \in \mathbb{Z}^d \colon T(0,z) \le t\} + \left[-\frac{1}{2}, \frac{1}{2}\right)^n.$$

What is the asymptotic growth as $t \to \infty$? It can be shown that $\frac{1}{n}T((0,0),(n,0))$ converges. We will show that (under some mild conditions) $\frac{1}{t}B(0,t)$ converges almost surely to a nonempty, compact, convex set B, which has the symmetries of $\mathbb{Z}^d$. In fact, for every $\varepsilon > 0$, there is a.s. a random time T such that for all $t > T$,

$$(1-\varepsilon)B \subset \frac{1}{t}B(0,t) \subset (1+\varepsilon)B.$$

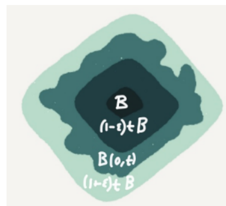
Is ∂B differentiable? Is B strictly convex?



January 6

Consider $\mathbb{Z}^2$ with an exp(1) distribution on edge weights. Recall that

$$B(0,t) = \{z \in \mathbb{Z}^2 \colon T(0,z) \leq t\}.$$



Theorem 1 (Shape Theorem)**.** *There exists a deterministic set B such that*

$$t(1-\varepsilon)B \subset B(0,t) \subset t(1+\varepsilon)B$$

for all sufficiently large t a.s.

Assume that the boundary of $B(0,t)$ looks like Brownian motion. Fix some $k \ll t^{2/3}$. Plot the points

$$\left(\frac{j}{k}, \frac{f(j)-f(0)}{k^{1/2}}\right), \qquad j = 0, \ldots, k$$

and interpolate to get a random function defined on the unit interval. Does this converge to Brownian motion? Based on simulations, it is conjectured that ∂B is not quite circular (but it has the symmetries of $\mathbb{Z}^2$).

Conjecture 1. *The curvature of ∂B is uniformly bounded above and below.*

This means that for all $x \in \partial B$, there are m, M such that for all $x \in \partial B$,

$$0 < m < \kappa(x) < M < \infty.$$

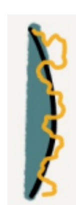
Conjecture 2 (Stronger version)**.** *B is strictly convex*

We are going to make a few assumptions today in order to heuristically explain where the magic exponent of $t^{2/3}$ is coming from.

Weakest assumption: boundary looks like Brownian motion.

Medium assumption: ∂B is a circle (more convenient form of the last assumption but it doesn't change much.)

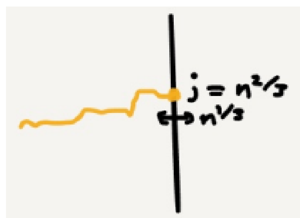
Strongest assumption: $f(j) = f(j) + S(j)$ where $(r(j), j)$ is on a circle $\partial(nB)$ where $S(j)$ is a simple random walk. (This is a way too strong assumption.)



Now consider $\max_{j\in\mathbb{Z}} f(j) - f(0)$ and $\operatorname{argmax} f(j)$. Suppose that $j = n^{\alpha}$ for some α. Then $|S(j)| \approx n^{\alpha/2}$. There are two competing things. Initially the SRW is the dominant term when j is small, but when j gets sufficienty large the distance you lose from the curvature will overpower anything you'll get from the random walk.

$$r(j) = \sqrt{n^2 - j^2} = \sqrt{n^2 - n^{2\alpha}} \approx n - \frac{1}{2n} n^{2\alpha}$$

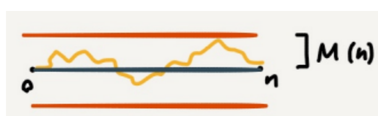
Then $f(j) \approx n - \frac{1}{2} n^{-1+2\alpha} + O(n^{\alpha/2})$. The question is when does $-1+2\alpha$ start dominating? That happens when $-1+2\alpha > \alpha/2$, which gives $\alpha > 2/3$.



We formulated this conjecture in terms of the fastest path from a point to a line. One may also state this in terms of the fastest path from a point to a point (perhaps this is more common).

Conjecture 3.

- $\operatorname{Var}(T((0,0),(n,0)) = n^{2/3+o(1)}$
- $M(n) = n^{2/3+o(1)}$



If you could answer these questions, then you would want to know the distribution of the time.

Conjecture 4 (Holy Grail)**.**

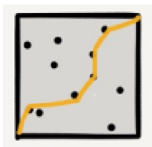
$$\frac{T((0,0),(n,0)) - \mathbb{E}T((0,0),(n,0))}{n^{1/3}} \to \textit{TracyWidom}$$

What is the Tracy Widom distribution? Let A be a Hermitian $n \times n$ matrix with $a_{ij} \in \mathbb{C}$ for $i > j$ independent complex Gaussians and a_{ii} are independent real Gaussians all with mean 0 and variance 1. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ be the eigenvalues (which are real). The Tracy-Widom distribution can be defined as

$$\lim_{n\to\infty} \mathcal{L}\left(\frac{\lambda_n - 2\sqrt{n}}{n^{1/3}}\right).$$

Two ways to look at this:

1. Longest increasing subsequence of a permutation
2. Last passage percolation



Convergence to Tracy Widom has been established. For the first one, take π a uniform random element of S_n (permutation on n elements). Then we want to know the length of the longest increasing subsequence

$$i_1 < i_2 < \cdots < i_k, \qquad \pi(i_1) < \pi(i_2) < \cdots < \pi(i_k),$$

say $H(n)$ is the largest such k. Then

$$\frac{H(n) - 2\sqrt{n}}{n^{1/3}} \to TW.$$

To get to last passage percolation, just plot $(i, \pi(i))$ in the square $\{1, \ldots, n\}^2$. The longest increasing subsequence is the longest northeast directed path from 1 to n and you count the number of dots you pass through. (There are two distributions where this is proven; exponential and one other. Everything is completely understood using the RSK algorithm.)

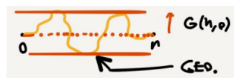
Outline: say what the main problems are (conclude on Friday). Then start talking about how you can show that this shape exists.

JANUARY 8

Consider first passage percolation on $\mathbb{Z}^2$. Last time we stated the conjectures

Conjecture.

- $\mathrm{Var}(T((0,0),(n,0)) = n^{2/3+o(1)}$
- $G(n,0) = n^{2/3+o(1)}$



Let $\text{GEO}(v,w)$ denote the geodesic from v to w in our random metric space. Define the **geodesic tree** to be $\mathbb{T} = \bigcup_{v\in\mathbb{Z}^2} \text{GEO}(v)$, where $\text{GEO}(v) := \text{GEO}(0,v)$.

How many ends does $\mathcal{T}$ have? At least one by compactness. By pigeonhole, at least one of $(\pm 1, 0)$ or $(0, \pm 1)$ is on $\text{GEO}(v)$ for infinitely many v. Repeating this argument produces an **infinite geodesic** (or an **end**), which is a sequence $(v_0, v_1, \ldots)$ such that for any $n < m$, the path $v_n, v_{n+1}, \ldots, v_m$ is the geodesic $\text{GEO}(v_n, v_m)$.

Conjecture 5. *There are uncountably many infinite geodesics, and at least one in every direction.*

More precisely, for every $x \in \mathbb{S}^1$ there exists some infinite geodesic $(v_0, v_1, \ldots)$ for which

$$\lim_{n\to\infty} \frac{v_n}{|v_n|} = x.$$

Theorem 2 (Hoffman)**.** *In any stationary percolation on $\mathbb{Z}^2$ satisfying mild assumptions, the geodesic tree has at least four ends.*

Consider a bi-infinite sequence $\ldots, v_{-1}, v_0, v_1, \ldots$. This is a geodesic if for any $n < m$ we have $\text{GEO}(v_n, v_m) = v_n, v_{n+1}, \ldots, v_m$.

Conjecture 6 (Hard)**.** *There is almost surely no bi-infinite geodesic.*

This is equivalent to a conjecture in the Ising model with random exchange constants. (Recall that the Ising model consists of $\{\pm 1\}^{\mathbb{Z}^2}$ with nearest neighbors corrrelated.)

The two big conjectures are the variance/fluctuations for the geodesic (and convergence to Tracy-Widom), as well as the conjecture about bi-infinite geodesics. They are all very intertwined. If you can prove that the fluctuations are bigger than $n^{1/2}$, then there is a heuristic argument allowing you to deduce the Ising conjecture.

Shape Theorem

We will talk about this for the next week or two, then move on to the variance.

$$B(0,t) = \{v \in \mathbb{Z}^2 \colon T((0,0),v) \le t\}$$

Theorem 3 (Shape Theorem). *There exists a convex non-empty set $B \subset \mathbb{R}^2$ that has all the symmetries of $\mathbb{Z}^2$ such that a.s.*

$$(1-\varepsilon)tB \subset B(0,t) \subset (1+\varepsilon)tB,$$

for all t sufficiently large. (We need a moment assumption about the passage times that will be discussed later.)

Before we get there let's talk about measure preserving transformations. (X,μ,T) where (X,μ) is a measure space, in fact a probability space $\mu(X)=1$. Then $T\colon X \to X$ is **measure preserving** if for all measurable $A \subset X$ we have $\mu(T^{-1}A) = \mu(A)$.

Recall that the passage times are a family of random variables $\{w(e)\}_{e \in E(\mathbb{Z}^2)}$. So our space is

$$X = (\mathbb{R}^+)^{E(\mathbb{Z}^2)}.$$

If the $w(e)$ are i.i.d., then consider the following transformation. For any $v \in \mathbb{Z}^2$ define

$$T_v \colon X \to X, \qquad (T_v(x))_e = X_{e+v}.$$

Moves the origin to v. Notice that we don't have just one measure preserving transormation, we have a whole family of them. So we actually have a **group action**. Our group is $\mathbb{Z}^2$ and for each $v \in \mathbb{Z}^2$ there is a measure preserving transformation T_v such that for all $v, w \in \mathbb{Z}^2$ we have $T_v(T_w(x)) = T_{v+w}(x)$.

Lemma 1. *Let T be an invertible measure preserving transformation such that T^{-1} is measurable. Then T^{-1} is measure preserving and we obtain a $\mathbb{Z}$ action with $T_i(x) = T^i(x)$.*

Sketch. If $\mu(TA) = \mu(A)$ then take $B = T^{-1}(A)$. □

Definition 1 (Ergodicity).

- *An m.p.t. T is* **ergodic** *if $\mu(A \Delta T^{-1}(A)) = 0 \implies \mu(A) = 0$.*
- *A group action is ergodic if*

$$\mu(T_g(A)\Delta A) = 0 \ \forall g \in G \implies \mu(A) \in \{0,1\}.$$

- *G is totally ergodic if T_g is ergodic for all $g \in G$.*

To see the difference between ergodicity and total ergodicity, consider the permutation action of S_2 on $\{1,2\}$: then the swap $(1\,2)$ is ergodic (w.r.t. uniform measure) and id is not ergodic.

We will show that $\{T_v\}$ is totally ergodic when μ arises from iid random variables.

JANUARY 11

Recap: Consider (X, T, μ) where μ is a probability measure on X and $T\colon X \to X$. We say that μ is invariant if $\mu(T^{-1}A) = \mu(A)$ for all measurable $A \subset X$. In this situation, T is **ergodic** if $\mu(A \Delta T^{-1}A) = 0$ implies $\mu(A) \in \{0, 1\}$ (i.e., 1_A is constant a.e.)

A key example to keep in mind is $X = (\mathbb{R}^+)^{E(\mathbb{Z}^2)}$ with $X(e) = w(e)$, where $w(e)$ are i.i.d. with a continuous distribution of finite expectation. Set $T_v(x)(e) = x(e+v)$, which is the map sending the origin to $v \in \mathbb{Z}^2$.

Claim 1. *Under these assumptions, T_v is ergodic for all $v \neq (0,0)$.*

Proof. Suppose not. Then there is some measurable $A \subset X$ with $0 < \mu(A) < 1$. First suppose that A is a cylinder set: it depends on only finitely many edges. Thus there is some n such that A and $T_v^{-n}A$ depend on disjoint sts of edges, and hence are independent. Therefore $\mu(A \cap T_v^{-n}A) = \mu(A)^2$. Now if A is invariant, we see that

$$|\mu(A \cap T_v^{-n}A) - \mu(A)| \leq \mu(A \Delta T^{-n}A) = 0,$$

so $\mu(A \cap T_v^{-n}A) = \mu(A)$ and therefore $\mu(A) = \mu(A)^2$ as desired. □

Suppose that (X, T, μ) is a measure preserving transformation and $f \in \mathcal{L}^1(X)$. The Birkhoff ergodic theorem states that for a.e. $x \in X$,

$$\frac{1}{n}\sum_{i=1}^{n} f(T^i X)$$

converges (this is called the **time average**).

Moreover if T is ergodic, the time average converges to the space average:

$$\int_X f \; d\mu.$$

There is an $\mathcal{L}^2$ version originally due to von Neumann, but the proof we present is due to Kakutani.

Theorem 4. *If (X, T, μ) is ergodic then the time averages converge in $\mathcal{L}^2$.*

Proof. Indeed, if $f(x) = f(T(x))$ a.e. then the space average is just $f(x)$ a.e. and the claim is immediate. Call these functions invariant and denote them by $\mathcal{M}$.

The next kind of functions for which this is clear are the cocycles $\mathcal{N}$, these are functions g of the form $g(x) = h(x) - h(T(x))$ a.e., for some function h. For these functions, we see that the time average telescopes:

$$\frac{1}{n}\sum_{i=1}^{n} g(T^i x) = \frac{h(Tx) - h(T^{n-1}x)}{n} \to 0.$$

Now it is not hard to check the decomposition $\mathcal{L}^2 = \mathcal{N} \oplus \mathcal{M}$, from which the theorem follows. □

There is also a $\mathbb{Z}^d$ version of the ergodic theorem. Let $\{T_v\}_{v\in\mathbb{Z}^d}$ be a family of measure preserving transformation of (X,μ) which commute: $T_v(T_w(x)) = T_w(T_v(x))$ for all $v,w \in \mathbb{Z}^d$ for all $x \in X$. If $\{T_v\}$ is ergodic (i.e. any set which is invariant under **all** the T_v has measure 0 or 1), then

$$\frac{1}{n^d}\sum_{1\le i_1,i_2,\ldots,i_d\le n} f(T_{e_1}^{i_1}T_{e_2}^{i_2}\cdots T_{e_d}^{i_d}(x)) \to \int_X f\ d\mu.$$

When is this useful? Take $f(x) = 1_{w(e)\in A}$ for some set $A \subset \mathbb{R}^d$. Then the ergodic theorem says that over some large cube, the fraction of (horizontal/vertical) edges in this big box where the weight $w(e) \in A$ is approximately $\mu(A)$ times the size of the box.

This is stronger than just the law of large numbers. Consider a percolation, with the event that a vertex is in the infinite cluster. Then this result allows us to conclude that the density of the infinite cluster in that box is approximately equal to the probability that the origin is in the infinite cluster. That event depends on infinitely many edges.

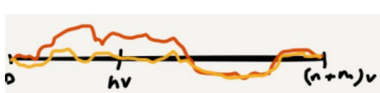
Another example: In first passage percolation, one can ask about the probability that there exists a bi-infinite geodesic. If this occurs with positive probability, then it has to go through the origin with positive probability (otherwise it would miss every vertex a.s., so by the union bound it doesn't exist).

There is a more general version of the $\mathbb{Z}^d$ ergodic theorem which applies to any amenable group; this is due to Lindenstrauss. (Remember, amenable groups are the groups on which you can do ergodic theory.) Følner sequences appear in place of cubes.

Kingman's subadditive ergodic theorem

Let (X,T,μ) be an ergodic measure preserving transformation. Let g_n be a sequence of $\mathcal{L}^1$ random variables satisfying $g_{m+n} \le g_n + g_m \circ T^n$. Then $\lim_{n\to\infty}\frac{g_n}{n} = \inf_{n\ge1}\frac{\mathbb{E}g_n}{n}$.

This theorem comes from percolation theory. Let (X,T,μ) come from an i.i.d. first passage percolation. Fix $v \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and let $g_n = T((0,0),nv)$. This satisfies the hypotheses of the subadditive ergodic theorem.



Assume that $g_n(x) \ge 0$ for all n and $x \in X$. Assume that T_v is totally ergodic and that T_v^n is ergodic for all n. To prove the theorem in this case, we need to find n such that

$$\frac{\mathbb{E}(g_n)}{n} < \ell + \varepsilon,$$

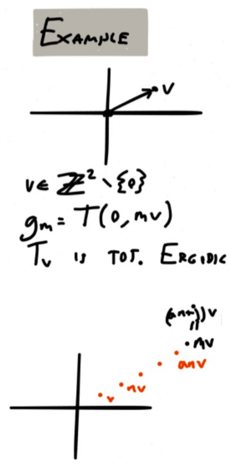
where $\ell = \inf\frac{\mathbb{E}(g_n)}{n}$. We leave this as a homework exercise.

January 13

Kingman's subadditive ergodic theorem

Let (X, T, μ) be a measure preserving transformation. Consider $\{g_n\}_{n\in\mathbb{N}}$ with $g_n \in \mathcal{L}^1$ such that $g_{m+n} \le g_n + g_m \circ T^n$. Let $\alpha = \inf \frac{\mathbb{E} g_n}{n}$. We will prove that $\lim_{n\to\infty} \frac{g_n}{n} = \alpha$ a.s.

Here is a picture of how this applies to first passage percolation:



Proof. Suppose that T is totally ergodic. Given any $\varepsilon > 0$, we want to find an n such that

$$\mathbb{E}\frac{g_n}{n} < \alpha + \varepsilon.$$

Write $m = an + j$ for $0 \le j < n$ (there is a unique such integer a). We want to show that $\frac{g_m}{m} < \alpha + 2\varepsilon$ a.s. for all m sufficiently large.

$$\begin{aligned} g_m &\le g_j + g_n \circ T^j + g_n \circ T^{j+n} \\ &= g_j + \sum_{i=0}^{a-1} g_n \circ T^{j+in} \\ &= g_j + \sum_{i=0}^{a-1} g_n \circ T^{in} \circ T^j \\ \implies \frac{g_m}{m} &\le \frac{g_j}{m} + \frac{1}{a}\sum_{i=0}^{a-1} \frac{g_n}{n} \circ T^{in} \circ T^j. \end{aligned}$$

By the Ergodic Theorem applied to the a points $x, Tx, \ldots, T^{a-1}x$, we see that the second term tends to $\mathbb{E}\frac{g_n}{n}$. There are finitely many j, so the first term vanishes in the limit $m \to \infty$. Therefore

$$\frac{g_m}{m} < \alpha + 2\varepsilon$$

for all m sufficiently large. Consequently $\lim_{m\to\infty} \frac{g_m}{m} \leq \alpha$ a.s. On the other hand, $\lim_{m\to\infty} \frac{g_m}{m}$ is a.s. constant and bounded below by α (perhaps by Fatou's lemma), yielding the claim. □

Shape theorem

Theorem. *There exists $B \subset \mathbb{R}^2$ such that*

$$t(1-\varepsilon)B \subset B_t((0,0)) \subset t(1+\varepsilon)B.$$

Consider the direction $v = (1,0)$. First, we claim that $\alpha > 0$ where

$$\alpha = \frac{\mathbb{E}T((0,0),(n,0))}{n}.$$

For any $\varepsilon > 0$, we can choose β such that $\mathbb{P}(w(e) < \beta) < \varepsilon$ since we have a continuous distribution.

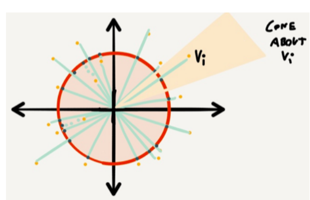
There are at most 4^n paths of length n from the origin; we're going to use the union bound to show that α is positive. Fix a path of length n. The probability it has $n/2$ edges of lenth $\leq \beta$ is at most

$$\binom{n}{n/2}\varepsilon^{n/2} \leq (2\sqrt{\varepsilon})^n.$$

Choose $2\sqrt{\varepsilon} < 1/4$ and apply the union bound to conclude. We have handled all directions simultaneously, and this shows that B has non-empty interior.

Note: the same argument is used in site percolation to show that there can't be too many bad events piling up, even though there are many possible arms. The criteria you need (in the case of a weight measure with atoms) is precisely that the mass at 0 is less than 1/2.

We want to choose unit vectors $v_1, \ldots, v_n \in \mathbb{Z}^2$ that are δn-dense in $\mathbb{S}^2$. More precisely, for all $(x,y) \in \mathbb{R}^2$ with $x^2+y^2 = n^2$ we want there to be some v_i for which $d(v_i,(x,y)) < \delta n$. On each of the rays nv_i, we will be able to apply the Ergodic Theorem.



Fix some v_i and ε. There is a δ such that for all sufficiently large v,

$$d\left(\frac{nv}{|v|}, v_i\right) < \delta.$$

In other words, v belongs to the δ-cone centered around v_i, and we will have

$$\left|\frac{T((0,0),v)}{|v|} - \alpha_i\right| < \varepsilon, \qquad \alpha_i = \lim_n T((0,0),nv_i)/n,$$

with ε to be specified shortly.

For each n, there are at most $(\delta n)^2$ many points. Moreover,

$$T((0,0), v) \leq T((0,0), nv_i) + T(nv_i, v).$$

For each such v, we have

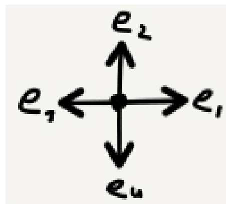
$$\mathbb{P}(\text{path from } nv_i \text{ to } v \text{ is longer than } 2\mathbb{E}w(e)\delta n) < c^n$$

for some $c < 1$ (by some large deviation inequality). So we can choose ε appropriately to complete the argument.

Note that existence of the first moment $\mathbb{E}w(e0)$ is stronger than what we need. If X_1, X_2 are iid, then we just want $\mathbb{E}\min(x_1, x_2) < \infty$. Roughly speaking, we can move in the diagonal direction at linear speed. We can also make things work for stationary distributions.

JANUARY 15

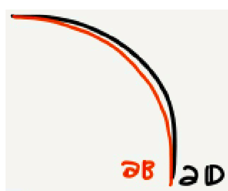
Last time we proved the shape theorem for i.i.d. edge weights with exponential tails. In general this works for stationary edge weights such that $\min_i w(e_i)$ has finite mean.



What do we know about B as a function of the distribution?

1. If $\mathbb{P}(w(e) = 0) < p_c(\mathbb{Z}^d)$ then B is not $\mathbb{R}^2$
2. If $\mathbb{E}w(e) < \infty$ then $\text{int}(B) \neq \varnothing$.
3. B has all symmetries of $\mathbb{Z}^d$
4. B is convex

Suppose the edge distribution is $C + \exp(1)$. Is it a circle? Or is it close to an $\mathcal{L}^1$ ball (i.e. diamond)?



Try to make a statement so weak that it can actually be proven. The shape varies continuously w.r.t. the distribution.

Theorem 5 (Auffinger-Damron)**.** *There exists a distribution such that B is not a polygon.*

There was an older theorem proving a conjecture of Kesten:

Theorem 6 (Damron-Hochman)**.** *For any k there is a distribution such that B has at least k extreme points.*

What about in high dimensions?

Theorem 7 (Kesten)**.** *If d is large ($d \geq 10^5$?) then a wide class of distributions have B not equal to the Euclidean ball (more like $\mathcal{L}^1$).*

The intuition is that you can move fast along the coordinate axes.

Theorem 8 (Haggstrom-Meester). *For any $B \subset \mathbb{R}^2$ satisfying 1-4 above, there exists a (relatively simple) stationary first passage percolation with B as the limit shape.*

There is a result providing partial information about how the shape changes when the distribution changes:

Theorem 9 (Van Den Berg-Kesten). *Suppose we have distributions F and $\widetilde{F}$ with F stochastically domainating $\widetilde{F}$, with $F \neq \widetilde{F}$ in distribution (and maybe some atom condition?). Then $\overline{B_{\widetilde{F}}} \subset \operatorname{int}(B_F)$.*

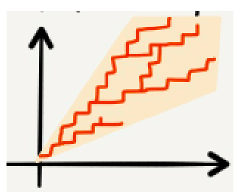
Directed percolation model

Let q be the smallest edge weight. More precisely,

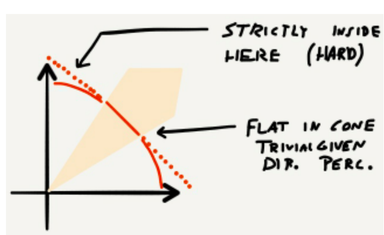
$$q = \sup\{q \colon \mathbb{P}(w(e) < q) = 0.$$

Suppose that $a = \mathbb{P}(w(e) = q) > p_c$(oriented percolation).

Any oriented path of edges with weight q forms a geodesic.
For any $a > p_c$(oriented), there is a cone that looks like the following:



It is centered on the line $y = x$ with cone angle depending on a such that the oriented percolation fills the cone, and hits finitely many points outside the cone. In first passage percolation, the limit shape has a portion on the $\mathcal{L}^1$ ball inside the cone and is strictly contained in the $\mathcal{L}^1$ ball outside the cone.



There is a conjecture which applies to the continuous case.

Conjecture 7. *If $w(e)$ has a continuous distribution then B is strictly convex.*

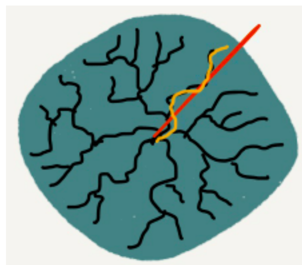
The result about infinitely many extreme points comes from studying a point on the edge of the cone. They show that the region is differentiable. There may also be a result about the second derivative, i.e. the curvature. This is the only result about the boundary of the shape.

January 20

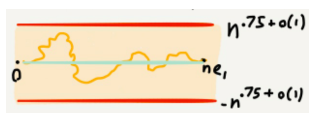
First we'll talk about hypotheses for what the shape looks like, then we'll talk about the variance.

Shape

If the shape is strictly convex, then there exists a geodesic in every direction.



If the limit shape has uniform curvature, then the geodesic from 0 to ne_1 has vertical fluctuations of order $n^{3/4+o(1)}$.



Conjecture 8. *If the edge weight distribution is continuous and has finite mean, then the shape is strictly convex.*

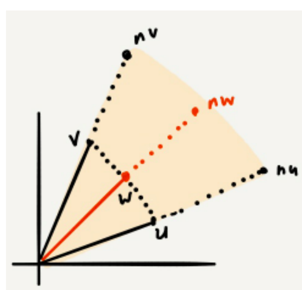


Figure 1: There are exponentially more paths of length L to nw than to either nu or to nv.

Fix some length $L = cn$ for $c > 1$ as $n \to \infty$. Let $F(c, v)$ be the number of paths from 0 to nv of length L. Then the total number of points is roughly $e^{f(c,v)n+o(1)}$ for some rate function $f(c, v)$. Moreover $f(c, v) > f(c, w) > f(c, u)$, for instance by estimating binomial coefficients.

Here are some hypotheses which can be used to make the previous description rigorous. It is not clear how to prove these or even that they are necessarily true. H1 is known to be true for spherically symmetric models, and it is believed to hold in general. H2 is believed to be false but is a simplification of what is believed to hold (the truth is believed to involved Tracy Widom scaling).

H1: For all convex cones $\mathcal{A}$, vectors $u \in \operatorname{Int} \mathcal{A}$ and all $\delta > 0$, there exists an $R > 0$ such that for all v near the line through v and the origin,

$$\mathbb{P}(\textsc{geo}(u, v) \subset \mathcal{A} \cup B(u, R)) > 1 - \delta.$$

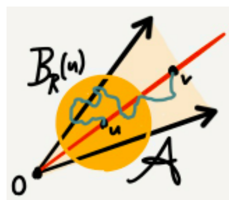


Figure 2: The geodesic only leaves $\mathcal{A}$ near u.

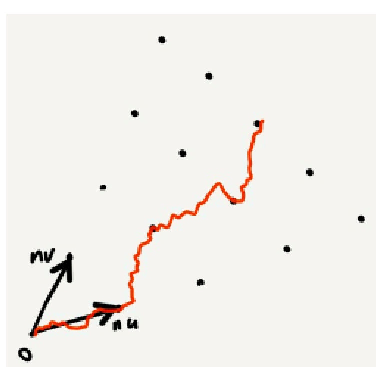
H2: For a fixed $u \in \mathbb{R}^2$, there exists a mean zero distribution G_u on $\mathbb{R}$ and $a(n)$ such that

$$\frac{T(0, nu) - n\mu(u)}{a(n)} \to_d G_u.$$

Here $\mu(u) = \lim_{n\to\infty} \mathbb{E}T(0, nu)/n$ is the inverse speed in the direction u. It is the norm whose unit ball is the limit shape in the shape theorem.

Theorem 10. *If we have continuous edge distributions and for $u, v \in \mathbb{R}2$ hypotheses H1 and H2 hold for u and v, then*

$$\mu(\lambda u + (1 - \lambda)v) < \lambda\mu(v) + (1 - \lambda)\mu(v), \qquad \forall \lambda \in (0, 1).$$



We reiterate that H1 is believed to be false for every reasonable distribution (i.e., in the correct universality class for Tracy-Widom to kick in). Since it is really useful to have a mean 0 distribution, we may ask if there are unreasonable distributions (e.g. heavy tails) that will produce a mean 0 distribution. Hard to say anything here though as the shape theorem won't hold.

Write $T(0,x) = \mu(x) + O(\|x\|)$, so that

$$O(\|x\|) = \underbrace{T(0,x) - \mathbb{E}(T(0,x))}_{\text{random fluctuation}} + \underbrace{\mathbb{E}\left[T(0,x) - \mu(x)\right]}_{\text{non-random fluctuation}} .$$

H2 can be reformulated as saying that the non-random fluctuations are smaller than the random fluctuations.

Variance

There is a function $\chi(x)$ such that under mild conditions, $\mathrm{Var}(T(0,x)) \sim \|x\|^{2\chi(d)}$.

d	$\chi(d)$	status
1	1/2	known
2	1/3	conjectured
3	no guess	

It is generally believed that $\chi(d) \to 0$ as $d \to \infty$. People have not worked too hard on this, however. This may be a reasonable problem to work on.

Theorem 11 (Kesten)**.** $0 \le \chi(d) \le 1/2$

The lower bound requires a short argument to show that the variance is at least a constant. Intuitively, the variance to get one step away from the origin is positive and bounded away from 0. In other words, if you resample the edges it will change the time by some constant amount. The upper bound will come from concentration inequalities.

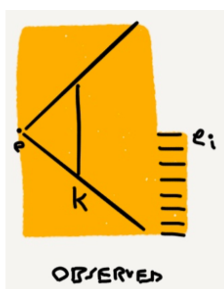
January 22

Conjecture 9. $\mathrm{Var}(T(0,x)) = \|x\|^{2/3+o(1)}$

We will show that $C < \mathrm{Var}(T(0,x)) < C\|x\|$ by putting it inside a big box. Enumerate the edges $e_1, e_2, \ldots, e_N$. Set $f(x) = T(0,x)$, $\Sigma_i = \sigma(w(e_1), \ldots, w(e_i))$ and $f_i = \mathbb{E}(f \mid \Sigma_i)$.

Write $f_i = f_{i-1} + Y$. Then $Y = \mathbb{P}(e_i \in \textsc{geo}(0,x) \mid \Sigma_{i-1})Z$ where $\mathrm{Var}\, Z = O(1)$ (here Z is close to the low end of the distribution of $w(e_i)$).

$$\begin{aligned}\mathrm{Var}(f) &= \mathbb{E}(f - \mathbb{E}f)^2 \\ &= \sum_{i=1}^{N} \mathbb{E}(f_i - \mathbb{E}(f \mid \Sigma_{i-1}))^2 \\ &\approx \sum_{i=1}^{N} \mathbb{P}(e_i \in \textsc{geo}(0,x) \mid \Sigma_{i-1})^2.\end{aligned}$$



We have an upper bound of order $O(1)$, and by Cauchy-Schwarz, we have a lower bound of order $1/n$ at distance n. This yields the bounds

$$c \log \|X\| \le \mathrm{Var}(T(0,x)) < C\|x\|.$$

Efron-Stein inequality

Let g be any function of the edge weights $w(e_1), \ldots, w(e_N)$. If we resample edge i to obtain $w'(e_i)$, let g_i denote g evaluated at this newly sampled value. Assume that the edge weights are independent.

Theorem 12 (Efron-Stein inequality)**.**

$$\operatorname{Var} g \leq \sum_{i=1}^{N} \left(\mathbb{E}(g - g_i)_+\right)^2$$

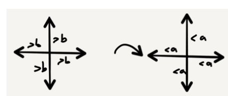
Heuristically, this says that the only things which affect the variance are the edges that lie on the geodesic.

There are two components of the Efron-Stein argument. First, we use the geometry to get a bound on the length of the geodesic. Then we use a concentration inequality.

Let Σ_0 denote the σ-algebra generated by all edges not incident to 0. Then

$$\operatorname{Var}(T(0,x) - \mathbb{E}(T(0,x) \mid \Sigma_0)) \leq \operatorname{Var}(T(0,x)).$$

Suppose that $w(e)$ is not concentrated on a single point. Then we can find constants a and b such that $\mathbb{P}(w(e) \leq a) > 0$ and $\mathbb{P}(w(e) \geq b) > 0$.



We get a constant lower bound on the variance, and using Efron-Stein yields a constant upper bound.

January 25

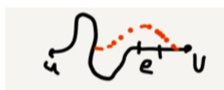
Last time we mostly showed that $C_1 < \mathrm{Var}(T(0,nx)) < C_2 n$. We would like to extend this to get deviation bounds.

$$\mathbb{P}(|T(0,nx) - \mathbb{E}(T(0,nx))| > t\sqrt{n}) < e^{-C_3 t^2}.$$

Consider first passage percolation on G and $n \times n$ torus. For $0 < a < b < \infty$ consider a distribution with $\mathbb{P}(w(e) = a) = \mathbb{P}(w(e) = b) = 1/2$. Set $f(w) = \mathrm{diam}_w(G)$. We want to show that $\mathrm{Var}(f) \leq Cn \log n$.

For any function f and edge e define $\sigma_e(f) = f(\widehat{w}_e)$ where $w(e') = \widehat{w}_e(e')$ for all $e \neq e'$ and $w(e) \neq \widehat{w}_e(e)$. Set $\rho_e f(w) = \frac{f(w) - \sigma_e(f(w))}{2}$. Consider f as before. If $\rho_e f(w) > 0$ then e is on every geodesic.

If $T(u,v) = f$ then there exists a geodesic from u to v containing e.



Given w, choose a triple (u,v,β) where β is a geodesic from u to v. Set $T(u,v) = f$ and observe that $T(u,v) \leq bn$. The number of edges in β is at most $(b/a)n$, so $\mathbb{E}|\beta| \leq Cn$ and

$$\sum_{e \in E} \mathbb{P}(e \in \beta) = \mathbb{E}(|\beta|)$$

imply that $\mathbb{P}(e \in \beta) \leq c/n$ (if everything is symmetric).

We want to argue that each edge has a small influence.

Theorem 13 (Talagrand)**.** *Let $f\colon \{a,b\}^J \to \mathbb{R}$ where J is a finite set. Then*

$$\mathrm{Var}\, f \leq C \sum_{j \in J} \frac{\|\rho_j f\|_2^2}{1 + \log(\|\rho_j f\|_2 / \|\rho_j f\|_1)}$$

Need to show $\|\rho_j f\|_2^2 \leq C/n$, then use Cauchy-Schwarz to conclude that the ratio of the 2 norm to the 1 norm is bounded below. Put it together to get the variance bound.

January 27

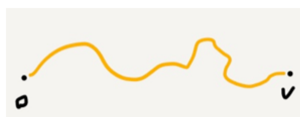
We are continuing to prove a result from last time.

$$\operatorname{Var}(f) \le C \sum_{j\in J} \frac{\|\rho_j f\|_2^2}{1+\log(\|\rho_j f\|_2/\|\rho_j f\|_1)}$$
$$\rho_j(f) = \frac{f(\omega)-f(\widetilde{\omega}_j)}{2}$$
$$\|\rho_j(f)\|_1 \le \|\rho_j(f)\|_2\sqrt{\mathbb{P}(\rho_j f \neq 0)}.$$

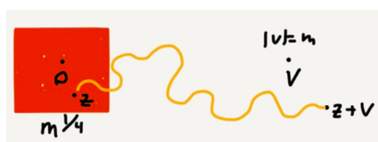
The last inequality follows from the following Cauchy-Schwarz:

$$\left|\rho_j(f)1_{\rho_j(f)\neq 0}\right| \le \|\rho_j(f)\|_2\|1_{\rho_j(f)\neq 0}\|_2.$$

Edges take the values a and b with equal probability.



Let $f = T(0,v)$ and consider $\widetilde{f}(z, z+v)$.



We have the crude bounds

$$\begin{aligned}|\widetilde{f} - f| &\le T(0,z) + T(v, z+v)\\ &\le Cm^{1/4}\end{aligned}$$
$$\operatorname{Var}(f) \le \operatorname{Var}(\widetilde{f}) + C_1 m1/4\sqrt{\operatorname{Var}(\widetilde{f})} + C_2 m^{1/2}.$$

There is a $g\colon [0,1]^{m^2} \to \{0,\dots,m\}$ such that $|g(\omega)-g(\widetilde{\omega}_a)| \le 1$ and $\max_y \mathbb{P}(g(\omega)=y) < C/m$. Let $S = (\{0,1\}^{m\times m})^d$. Define $z(s) = (g(s_1),\dots,g(s_{m\times m}))$, $s = (s_1,\dots,s_{m\times m})$ where $s_i \in \{0,1\}^{m\times n}$.

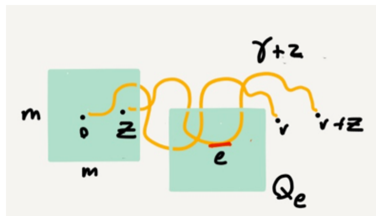
$$\max_y \mathbb{P}(z=y) \le \left(\frac{C}{m}\right)^2.$$

$f = T(0,v) \le b|v|_1$ and $\widetilde{f} = T(z, z+v)$

γ is a geodesic from 0 to v chosen deterministically from all geodesics.

$\mathbb{P}(p_e(f) \neq 0) = 2\mathbb{P}(p_e(f) > 0)$. Suppose changing e lowers f. Then $w_e = b$ and e was on at least one geodesic. If changing e raises f, then $w_e = a$ and e is on every geodesic. Thus it is on γ. At most $(b/a)|v|_1$ edges raise f. $\mathbb{P}(p_e(\widetilde{f}) \neq 0)$.

Condition on γ. We want to know $\mathbb{P}(e \in \gamma + z \mid \gamma)$.



If $e \in \gamma + z$ then there is an e' in $e + [0, m]^2$ on γ. There are at most $2(b/a)m$ edges in $\gamma \cap Q_e$. Therefore

$$\mathbb{P}(e \in \gamma + z \mid \gamma) \leq \sum_{e' \in Q_e} \mathbb{P}(e' \in \gamma \mid \gamma)\mathbb{P}(z = e - e') = C/m.$$

This yields $\sum_{e \in E} p_e(\widetilde{f}) \leq 2(b/a)|v|_1$.

$$\begin{aligned} &\sum_{e \in E} \frac{\|p_e(\widetilde{f})\|_2^2}{1 + \|p_e(\widetilde{f})\|_2 / \|p_e(\widetilde{f})\|_1} \\ &\leq \sum_{e \in E} \frac{\|p_e \widetilde{f}\|_2^2}{C \log |v|_1} \\ &\leq \frac{C|v|_1}{\log |v|_1}. \end{aligned}$$

For the other term,

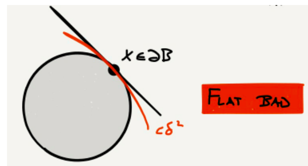
$$\sum_{s \in S} p_s(\widetilde{f}).$$

Now $|S| = 2m^2 = 2|v|^{1/2}$ and the geodesic changes by at most $b - a$. Now we bound the influence by working with the numerator and ignoring the denominator.

January 29

Variance bounds yield fluctuation bounds.

1. $\text{Var}(T(0, x)) < C\|x\|$
2. Uniformly bounded curvature



3. Concentration bound

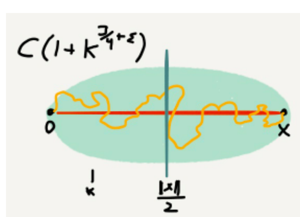
$$\mathbb{P}\left(|T(0, x) - \mathbb{E}T(0, x)| > \lambda\|x\|^{1/2}\right) < Ce^{-c\lambda}$$

Consider $\text{GEO}(0, x)$. From now on whenever we know something about the variance, we have a concentration inequality as well (they are coming from the same place).

1. $\text{Var}(T(0, x)) < C\|x\|$
2. uniformly bounded curvature
3. concentration inequalities

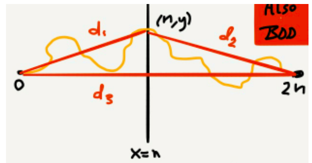
Regarding curvature we can say more. Put a parabola in from the tangent (assume there is a unique tangent). For $x \in \partial B$ we put a parabola $c\delta^2$. Assume that this ball is actually circular (the flat parts cause problems). The expected distance is Euclidean.

Theorem 14 (Piza, Newman)**.** *If the shape is uniformly curved, we have concentration inequalities, and* $\text{Var}(T(0, x)) < C\|x\|$*, then for* $\|x\|$ *the geodesic* $\text{GEO}(0, x)$ *fluctuates no more than* $Ck^{3/4+o(1)}$ *at distance* k *along the straight line from* 0 *to* x*.*



If you have a beter variance bound, you get a better fluctuation bound. In general if the variance is on the order of n^α then the fluctuations are on the order of $n^{1/2+\alpha/2}$. Thus if we knew the variance grew like $n^{2/3}$ we would get the 'correct' fluctuations $n^{2/3}$.

We will assume that $\mathbb{E}(T(0, x)) = \|x\|_2$ to keep things simple.



Next we want an upper bound on $\mathbb{P}((n, y) \in \text{GEO}((0, 0), (2n, 0))$. Let $d_1 = d_2 = \sqrt{n^2 + y^2} \approx n + y^2/2n$ and $d_3 = 2n$. Then $d_1 + d_2 - d_3 \approx y^2/n$. If $(n, y) \in \text{GEO}$ then $T_1 + T_2 = T_3$, where

$$\begin{aligned} T_1 &= T((0,0),(n,y)) \\ T_2 &= T((n,y),(2n,0)) \\ T_3 &= T((0,0),(2n,0)). \end{aligned}$$

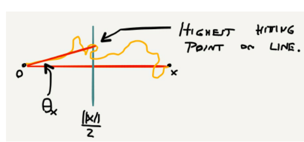
If $T_1 + T_2 = T_3$ then at least one of the following happens:

1. $T_1 \leq d_1 - y^2/3n$
2. $T_2 \leq d_2 - y^2/3n$
3. $T_3 \geq d_3 + y^2/3n$

Let $\lambda = (y^2/3n)/\sqrt{n}$ be the 'number of approximate standard deviations' away. If $y > n^{3/4+\varepsilon}$ then $\lambda > n^{2\varepsilon}/3$. So by concentration this probability occurs eiwith probability less than $e^{-Cn^{2\varepsilon}/3}$.

Summing over all $|y| > n^{3/4+\varepsilon}$ gives a rapidly decaying tail that grows like $n^{2\varepsilon}$. Therefore we get

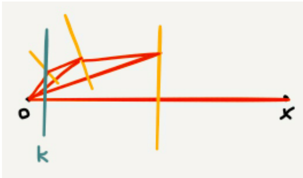
$$\mathbb{P}(\exists y \colon |y| > n^{3/4+\varepsilon} \text{ and } (n, y) \in \text{GEO}((0, 0), (n, 0))) \leq Ce^{-cn^{2\varepsilon}}.$$



With high probability, $\theta_x \leq n^{-1/4+o(1)}$. Taking a union bound over x on the circle of radius n gives

$$\mathbb{P}(\exists x \colon \theta_x > Cn^{-1/4+\varepsilon}) < C'e^{-c'n^{\varepsilon}}.$$

Consider points a sequence of points y_n with $\|y_n\| = 2^{n-1}\|y_1\|$. We have bounds $\theta_{y_n} \leq C(2^{n-1}\|y_1\|)^{-1/4+\varepsilon}$.



Summing over the angles gives an upper bound on the cone for the final geodesic:

$$\sum_{n=1}^{N} \theta_{y_n} = \|y_1\|^{-1/4+\varepsilon} \sum_{n=1}^{N} (2^{n-1})^{-1/4+\varepsilon},$$

which is a convergent sum. Translating this back to the earlier picture gives us the $k^{3/4+\varepsilon}$ bound.

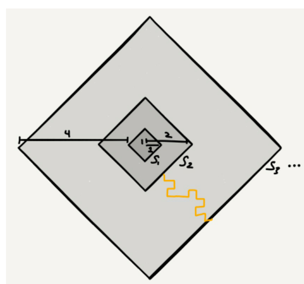
Note that this would tell you that there is an infinite geodesic in every direction. Moreover if you were working in a spherically symmetric model, then you would also get unique geodesics in every direction.

FEBRUARY 1

Label edges e_i. Create σ-algebras $\Sigma_i = \sigma(\omega(e_1), \dots, \omega(e_i))$. Let $f = T((0,0), x)$ and $f_0 = \mathbb{E}f = \mathbb{E}(f \mid \Sigma_0)$ and $f_i = \mathbb{E}(f \mid \Sigma_i)$.

Then $\operatorname{Var} f = \mathbb{E}(f^2) - \mathbb{E}(f)^2 = \sum_0^{N-1} \mathbb{E}[(f_{i+1} - f_i)^2]$. Let A_i be the edge e_i is in the geodesic and $w(e_i) > a$. With probability p e_{i+1} on the geodesic. With probability $p/2$ increases it by 1, with $p/2$ decrease by 1. So $\mathbb{E}(f_{i+1} - f_i)^2 = p^2$ heuristically (to be precise, it will depend on the distribution).

So $\operatorname{Var} f \approx C \sum_0^{N-1} \mathbb{P}(A_{i+1} \mid \Sigma_i)^2$. Now recall that $\mathbb{E}(\#A_i) = cn$. In two dimensions this will give us a lower bound on the variance by Cauchy-Schwarz. Draw a bunch of exponentially increasing concentric annuli. Any geodesic from 0 to x has at least 2^k edges between S_k and S_{k+1}. Choose a to be a small number (continuous distribution). What's the probability that there exists a path connecting S_k to S_{k+1} with at least 1/2 the number of edges with weight $\leq a$?



Between S_k and S_{k+1} there are $c4^k$ edges (since we're in $d = 2$). For $c'2^k$ of them, the event A_i occurs. By Cauchy-Schwarz, the variance is minimized when the A_i events are roughly uniformly distributed. After cancellation we get a constant lower bound from uniform. There are $\log n$ annuli, so the variance is at least $\log n$.

Unconditional lower bounds on fluctuations

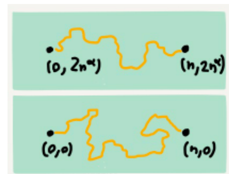
Define an exponent

$$\xi^\circ = \sup\{\alpha > 0\colon \limsup_{\|x\| \to \infty} \mathbb{P}(\textsc{geo}(0,x) \subset C(x, x^\alpha) < 1\}$$

This is the largest number so that there is a reasonable change that the geodesic from 0 to x will go outside the region. For any α bigger than this, the geodesic is constrained in the region. Here $C(x, \|x\|^\alpha)$ is more or less the thickening of $[0, x]$ by $\|x\|^\alpha$. For all $\alpha > \xi^0$ the geodesic is contained in this cylinder $C(x, \|x\|^\alpha)$. We want to show that ξ° cannot be small, specifically that $\xi^\circ \geq 1/3$.

Theorem 15 (Newman, Piza)**.** $\xi^\circ \geq 1/3$

Proof. Let $f = T((0,0),(n,0) - T((0,2n^\alpha),(n,2n^\alpha))$.



We want to calculate its variance. Looking at the difference, we get $|f| \leq T((0,0),(0,2n^\alpha)) + T((n,0),(n,2n^\alpha)) \leq Cn^\alpha$. But we can rerun our old argument to get a contradiction when α is too small. □

FEBRUARY 3

Last time we lower bounded the variance of the function $f = T(0, v)$ as follows. Label the edges e_i and set $\Sigma_i = \sigma(\omega(e_1), \ldots, \omega(e_N))$. Let A_i be events that 'depend' on e_i: they change the value of f by a constant.

Then we use

$$\operatorname{Var} f \geq \sum_i \mathbb{P}(A_{i+1} \mid \Sigma_i)^2.$$

Given the value of

$$A = \sum_i \mathbb{P}(A_i) = \mathbb{E}(\# \text{ of } i \text{ s.t. } A_i \text{ occurs}),$$

we can lower bound the previous expression using Cauchy-Schwarz: the minimum is achieved by distributing the mass uniformly, i.e. all probabilities are A/B where B is the number of edges that B depends on.

Use this technique to get a lower bound of $C \log n$ for $\operatorname{Var}((0,0),(n,0))$ on $\mathbb{Z}^2$. (We get only a constant lower bound in higher dimensions, which is expected to be wrong but we don't know.)

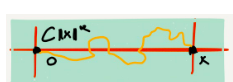
Now we apply this to fluctuations.

$T_1 = T((0,0),(n,0))$ and $T_2 = T((0,2n^\alpha),(n,2n^\alpha))$ and consider the function $f = T_1 - T_2$. Then by the triangle inequality

$$\begin{aligned}|f| &\leq T((0,0),(0,2n^\alpha)) + T((n,0),(n,2n^\alpha)) \\ &\leq Cn^\alpha.\end{aligned}$$

Recall the arm exponent we defined last time, called ξ°, defined such that for all $\alpha > \xi^\circ$ we have

$$\lim_{\|x\| \to \infty} \mathbb{P}(\text{GEO}((0,x)) \subset C(x, \|x\|^\alpha)) = 1.$$



(In other words, the geodesic stays within $\|x\|^\alpha$ of the line connecting 0 to x.) If α is small, then the technique from last time gives a lower bound on the variance bigger than Cn^α thereby obtaining a contradiction.

Let $A_i = \{e_i \text{ is on lower geodesic, not on upper geodesic, and } \omega(e_i) > \lambda > 0\}$. In other words, if you lowered it by a certain amount it would still be on this geodesic. (At this point, we are implicitly working with distributions that don't have an atom at the bottom of their support. Apparently there are technical ways to fix this that we won't worry about.)

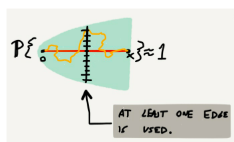
From the definition of the exponent α, with high probability the geodesic stays in the box. Therefore $\mathbb{E}(\#\text{ of } A_i \text{ which occur with } e_i \notin C((n,0),n^\alpha)) \geq Cn$. But $|C((n,0),n^\alpha)| \approx Cn \cdot n^\alpha$, so we get the bound

$$\operatorname{Var} f \geq (c'n)^2/(cn^{1+\alpha}) \approx n^{1-\alpha}.$$

Since $\operatorname{Var} f \leq f^2$ and $|f| \leq Cn^\alpha$ with high probability, we have the upper bound $\operatorname{Var} f \leq Cn^{2\alpha}$. Thus $2\alpha \geq 1 - \alpha$ so $\alpha \geq 1/3$.

Spherically symmetric model

Recall from last Friday that with high probability, the geodesic stays within $k^{3/4+o(1)}$.



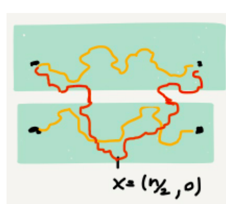
Let's pretend that any e_i on the geodesic satisfies $\omega(e_i) > \lambda$. The variance of the passage time is lower bounded by

$$\operatorname{Var} T((0,0),(x,0)) \geq \sum_{k=1}^{n} \frac{k^{3/4+o(1)}}{k^{3/4+o(1)}} \approx n^{1/4}.$$

Now we improve the fluctuation bound. Use the same lower bound as before:

$$n^{1-\alpha+o(1)} \leq \operatorname{Var} f.$$

But for the upper bound, we can cut diagonally across.

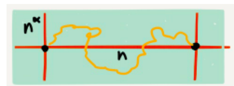


The Euclidean distance from $(0, n^\alpha)$ to x is

$$\sqrt{(n/2)^2 + (Cn^\alpha)^2} \approx n/2 + 2/nC^2n^{2\alpha} \approx n/2 + cn^{2\alpha-1}.$$

February 5

In the case of uniform curvature, $\alpha > \xi^\circ = 3/5$.



Infinite geodesics

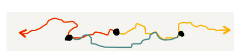
How many ends does the tree of one-sided infinite geodesics from 0 have?

General belief: For an i.i.d. continuous distribution, for every direction $v \in S^1$ there exists a unique geodesic $G = 0 = g_0, g_1, \ldots$ and with $g_i/\|g_i\| \to v$ a.s.

These results are true if the limiting shape has bounded curvature. This is because in the radial model, geodesics didn't deviate far from straight lines, so this says that there are geodesics in every direction.

It is known that for any i.i.d. continuous model (and many stationary distributions) there are at least 4 geodesics. Can weaken the curvature assumption to strictly convex.

Conjecture 10. *No bi-infinite geodesics a.s., because as you go backwards in time there are always better and better shortcuts that don't use the same path.*



It is known that:

1. In any fixed half plane, there are no bi-infinite geodesics a.s.

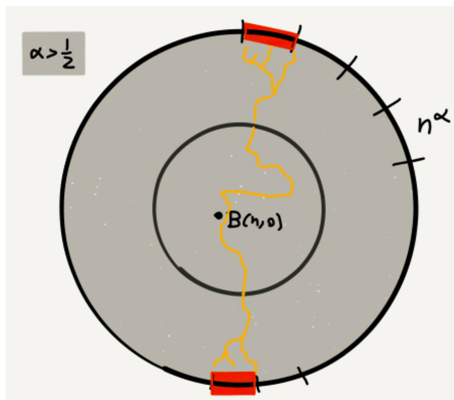
2. For a.e. fixed direction v there is a.s. no $(v, -v)$ bi-infinite geodesic.

Suppose the fluctuation exponent $\xi^\circ > 1/2$ (with no uniform curvature assumption). Also make a stronger assumption on a coalescence exponent.

Consider the largest α such that this has probability bounded away from 0 of occuring. Suppose the coalescence exponent is bigger than 1/2 uniform over all directions.

It is clear that the fluctuation exponent is at least as large as the coalescence exponent, because if the geodesics meet then one of them had to fluctuate by at least n^{α}.

Going from a large fluctuation exponent to a large coalescence exponent would probably be enough to show the non-existence of bi-infinite geodesics.



Number of intervals is on the order of $n^{1-\alpha}$. For all pairs of intervals, all geodesics between two poits agree in $B(n/3, 0)$. Inside the vall there are $\sim cn^2$ vertices. For every pair of intervals there are $< C'n$ vertices on a geodesic, so there are a total of $n^{2-2\alpha}$ pairs of intervals.

If both conditions on the previous board hold, then there are $O(n^{2-2\alpha} \cdot n$ vertices on a geodesic connecting two points on the large circle.

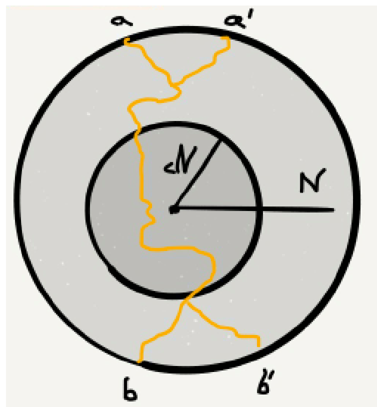
Now we need to relax our assumptions. The problematic assumption is that

You can define an equivalence relation, where two pairs of points on the boundary are in the same equivalence class if their geodesics agree in the circle. Really we want to say that the number of equivalence classes is less than n.

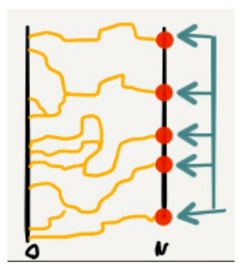
If you can get enough coalescence to happen, then you can prove the non-existence of bi-infinite geodesics. There don't seem to be results about coalescence exponents, but they should be roughly the same as the fluctuation exponents.

Recap of argument: we want to show that there are no bi-infinite geodesics. By the union bound, it suffices to show that there is no bi-infinite geodesic through the origin with positive probability. If a point is on a bi-infinite geodesic then it must be on a geodesic connecting points on the boundary. Now we take the union bound over all the pairs of equivalence classes of geodesics.

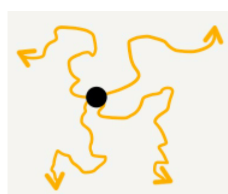
February 8



If the number of equivalence classes is $o(N)$ then there is no bi-infinite geodesic almost surely. Recall the coaelescence exponent, related to the point to line geodesic. Suppose the size of a typical equivalence class is $N^{1/2+o(1)}$.



Also consider one-sided geodesics.



Let X_0 be the number of one-sided geodesics starting at 0. By compactness $X_0 \geq 1$.

Theorem 16 (Haggstrom-Pemantle). $\mathbb{P}(X_0 > 1) > 0$

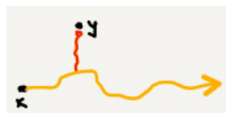
When edge weights are i.i.d. exponential on $\mathbb{Z}^2$. By the end of this week we will show $\mathbb{P}(X_0 \geq 4) = 1$.

Let A be the event every vertex has a unique geodesic. They all coalesce. We want to show that $\mathbb{P}(A) = 0$ for a wide class of edge weight distributions.

Assume that $X_v = 1$ for all v. Then two points are equivalent if their geodesics coalesce. Look on the boundary (with say a continuous distribution on edge weights) then you're going to get multiple geodesics from one point.

Two geodesics coalesce if the symmetric difference is finite.

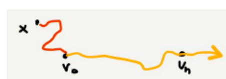
Consider any geodesic $G = (v_0, v_1, \ldots)$. It defines a Buseman function, which is a measure of how far from infinite two points are in a given direction. For example in Euclidean space, it is just the orthogonal projection on a direction towards infinity.



It is defined as

$$B_G(x,y) = \lim_{n\to\infty} T(x,v_n) - T(y,v_n).$$

Lemma 2. *For any one-sided geodesic G and any x, y the Busemann function exists.*

Proof. 

Consider the special case $y = v_0$. Let $a_n = T(x,v_n) - T(v_0,v_n)$. By the triangle inequality, $a_n \le T(x,v_0) = a_0$. The same argument shows that a_n is non-increasing. We also have a symmetrical lower bound, so we have convrgence.

Writing out $B_G(x,y)$ as a combination of a_n and $b_n = T(v_0,v_n) - T(y,v_n)$ yields convergence of the Buseman function. Moreover by the triangle inequality, $|B_G(x,y)| \le T(x,y)$.

If A occurs, then there is a unique geodesic to form the Buseman function with. Of course it depends on the edge configuration ω. Now consider shifts of the edge configuation; then $B^\omega(x,y) = B\sigma_v(\omega)(x-v, y-v)$.

Properties of $B_G(x,y)$:

1. $|B_G(x,y)| \le T(x,y)$ with equality if x is on the geodesic from y or vice versa.
2. Shift invariance
3. Additivity

□

By the latter two properties,

$$\begin{aligned} B^\omega((0,0),(n,0)) &= \sum_{i=0}^{n-1} B^\omega((i,0),(i+1,0)) \\ &= \sum_{i=0}^{n-1} B^{\sigma_{i(,0)}\omega}((0,0),(1,0)). \end{aligned}$$

Recall that we're considering a stationary edge weight distribution. Then

$$\frac{B^\omega((0,0),(n,0))}{n} = \frac{1}{n}\sum_{i=0}^{n-1} B^{\sigma_{i(,0)}\omega}((0,0),(1,0)).$$

As $n \to \infty$, ergodic theory implies that

$$\lim_{n\to\infty} \frac{1}{n} B^\omega((0,0),(n,0)) = \mathbb{E}B((0,0),(n,0)).$$

For this to work, we assume that $\omega(e)$ has finite mean to get that the Buseman function has finite moment. Assuming lattice symmetry, we see that the last expectation is zero.

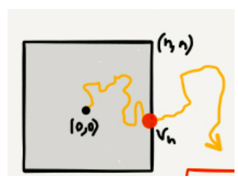
FEBRUARY 10

Assumptions:

1. $\mathbb{E}\omega(e) < \infty$
2. $\{w(e)\}$ is totally ergodic
3. $\{w(e)\}$ has all the symmetries of $\mathbb{Z}^d$

(We consider $d = 2$ for notational convenience mostly.)

Consider the event $A = \{$every vertex has a unique geodesic and these all coalesce$\}$. $\mathbb{P}(A) = 0$. If A then $B(x, y) = \lim_n T(x, v_n) - T(y, v_n)$ where v_i is the geodesic away from some vertex. The busemann function satisfies $B(x, y) + B(y, z) = B(x, z)$ and $|B(x, y)| \le T(x, y)$. Equality if x is on geodesic from y. $\mathbb{E}|B((0, 0), (1, 0))| < \infty$. By symmetry $\mathbb{E}B((0, 0), (1, 0)) = 0$ and similarly $\mathbb{E}B(x, y) = 0$ for all $x, y \in \mathbb{Z}^2$.



$B(0, v_n) = T(0, v_n) \ge cn(1 + o(1))$ for some $c > 0$. On the other hand

$$B((0,0),(n,0)) = \sum_{i=0}^{n-1} B^{\sigma_?(w)}((0,0),(1,0)),$$

which divided by n tends to $\mathbb{E}B((0, 0), (1, 0)) = 0$.

Moreover this argument works for finitely many points simultaneously. Take a bunch of points spaced out by εn.

Use the shape theorem to get c, choose $\varepsilon = c/10$. Choose finitely many points $\{v_i\}_{i=1}^k$ on the unit square, which are $\varepsilon/10\mathbb{E}\omega(e)$ dense.

By the ergodic theorem, if n is sufficiently large then with probability at least $1 - 1/3k$ we have $|B(0, nv_i)| < \varepsilon n$. By the shape theorem, with probability $1 - 1/3k$ the ball of radius εn around the origin in the fpp norm contains all points on the axis of distance at most $\varepsilon \mathbb{E}\omega(e)n/2$. Thus with high probability for all v on the square, $|B((0, 0), v)| < \varepsilon n$.

Where did the idea of Busemann functions come from? In a paper of Newman. Consider points at $(0, 0)$ and $(n, 0)$ and let

$$S_n = \{v \in \mathbb{Z}^2 \colon T(v, (n, 0)) \le T((0, 0), (n, 0)\}.$$

When you take n to ∞, you may hope that the S_n sets convergence to get some kind of half-plane. The boundary between the limit of the sets and its complement is a level set of the Busemann function. They are useful because they are linear functionals which allow us to tell geodesics apart: it gives us a precise way to tell when they have different limiting slopes.

Set $B(x, y) = \lim_{n\to\infty} T(x, L_n) - T(y, L_n)$ where $L_n = \{(n, \star)\}$. Suppose that the geodesics from x to L_n eventually start the same way (i.e. they converge). Call the resulting geodesic G_x. Also suppose that for al $x, y \in \mathbb{Z}^2$, the symmetric difference of G_x and G_y is finite.

If this happens then we get a well-defined Busemann function. Moreover by symmetry $\mathbb{E}B_n((0,0),(0,1)) = 0$. Also $|B_n((0,0),(0,1))| \leq \tau((0,0),(0,1))$ which is in $\mathcal{L}^1$. Since the geodesics converge, the Busemann functions also converge:

$$B_n((0,0),(0,1)) \to B((0,0),(0,1)).$$

By Dominated Convergence, $\mathbb{E}B((0,0),(0,1)) = 0$ as well. On the other hand, $\mathbb{E}B((0,0),(1,0)) > 0$ (more work here). If the geodesic convergence condition held, then we would get infinitely many different geodesics.

But we can still get an interesting result even without the geodesics converging. Suppose that the shape is differentiable where it meets the x-axis. Then we can construct a random geodesic such that its Busemann function B has

$$\mathbb{E}B((0,0),(0,1)) = 0, \qquad \mathbb{E}B((0,0),(1,0)) > 0.$$

Theorem 17. *There are at least* 4 *tangent points at which the boundary is differentiable, and hence there exist at least* 4 *geodesics with different Busemann functions.*

February 12

There will be presentations during the last week of class in groups of two. Choose one of the following papers/topics:

1. Haggstrom-Meester: Any shape is possible for stationary fpp.
2. Wehr-Woo: No bigeodesics in a half plane
3. Kesten-Auffinger-S: Shape in higher dimensions.

Assumptions on edge-weights:

1. shift invariant
2. ergodic
3. $\mathbb{E}w(e)^{2+\delta} < \infty$ for some $\delta > 0$
4. limit shape non-trivial
5. all symmetries of $\mathbb{Z}^2$
6. unique geodesics
7. finite upward energy

Finite upward energy means the following.

Condition on all edge weights except e_0. The distribution of $w(e_0)$ given $\{w(e_i)\}_{i \neq e_0}$ is Σ. For all $\lambda > 0$ if $\mathbb{P}(w(e_0) > \lambda \mid \Sigma) > 0$, then there is a $\lambda' > \lambda$ such that $\mathbb{P}(w(e_0) > \lambda' \mid \Sigma) > 0$.

Any two sets of energies inside A and outside

Anything possive outside pos inside A has positive probability

If can happen on one collection and the other, both positive probability.

No mass on largest value. If continuous then good shape. No atom at its ess sup.

Suppose we have all properties listed above. Consider any cone in boundary of shape...

Intersection of limiting shape and cone are a triangle. $\{x_i\}_{i\in\mathbb{N}}$. Then the direction of x_i is all limit points of $x_i/\|x_i\|_2$. Such a cone is a generalized direction. If the x_i lie along a path, then the limit points form a connected subset of $\mathbb{S}^1$ (because the image of a connected set under a continuous map is connected).

Call such a cone a generalized direciton.

Theorem 18.

Existence For any generalized direction d and for all $x \in \mathbb{Z}^2$ there is at least one geodesic G_x with direction contained in d.

Coalescence For all $x, y \in \mathbb{Z}^2$, there are geodesics G_x and G_y starting at x and y with direction contained in d that coalesce.

Backwards finite For all x, the set of z such that $x \in G_z$ is finite.

Let $\Omega_1 = (\mathbb{R}^+)^{E(\mathbb{Z}^2)}$. Enlarge the space. Fix a line $L_n = \{(x, y) \colon x = n\}$. Consider $\text{GEO}(x, L_n)$ and the Busemann function $T(x, L_n) - T(y, L_n)$. Since the Busemann function is linear, we only need to know its values on any directed edge e. Thus from $\{B_n(e)\}_{e \text{ directed}}$ we can reconstruct the function $B_n(x, y)$ for all $x, y \in \mathbb{Z}^2$. If you know this information, you can determine all the geodesics.

For each directed edge, is there an $x \in \mathbb{Z}^2$ such that this directed edge is on the geodesic from x to L_n? This is $f_n(e)$.

Keep track of $\omega(e)$ for all $e \in E(\mathbb{Z}^2)$, also $B_n(e)$ and $f_n(e)$. Let $\Omega_2 = (\mathbb{R}^+)^{\text{directed edges}}$ and $\Omega_3 = \{0, 1\}^{\text{directed edges}}$. The entire space $\Theta = \Omega_1 \times \Omega_2 \times \Omega_3$. We will construct a shift-invariant measure on this space such that every point in the support is consistent and satisfies the conclusions of the theorem.

For every n, we have a measure m_n on Θ built out of geodesics to the line L_n. We claim that this sequence of measures has a subsequential limit. If the edge sets were bounded and we replaced $\mathbb{R}^+$ with a compact set, then this would be a compactness result. To get around this technical difficulty, consider a wifxed ω_1. Then $m_n \restriction \Omega_2 \times \Omega_3$ has compact support, because the Busemann function of any directed edge is bounded by the edge weight, which is finite (i.e. there is no mass at edge weight infinity).

To show shift invariance, first notice that there is no problem with vertical shifts. But horizontal shifts change the line we are heading towards, so instead we consider $\widetilde{m}_n = \frac{1}{n} \sum_{i=1}^{n} m_i$. This is the fraction of times that x is on one of those lines. As n gets large, this becomes more and more invariant under horizontal shifts.

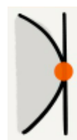
Take a point $\omega_1 \in \Omega_1$. Put a dirac measure on $(\omega_1, \{B_n(e)\}, \{f_n(e)\})$ corresponding to the unique valid configuration. Then the average is more smeared out so it can be non-deterministic.

February 17

Talk schedule:
Week 9 Friday: Gerandy and Clayton
Week 10 Monday: Chris F and Shirshendu (High dimensions)
Week 10 Wednesday: Avi and Peter (No two sided geodesics)
Week 10 Friday: Jacob and Yizhe (Acheiving different shapes)

Back to the math

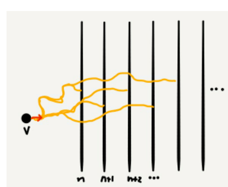
Boundary of shape is differentiable at rightmost point.



$L_n = \{x = n\}$. $\widetilde{\Omega} = \Omega_1 \times \Omega_2 \times \Omega_3$. The space Ω_1 represents edge weights $\mathbb{R}^{E(\mathbb{Z}^2)}$, Ω_2 represents a direction at every vertex $4^{\mathbb{Z}^2}$, and Ω_3 represents a Busemann function $B(0, e_1)$ and $B(0, e_2)$.

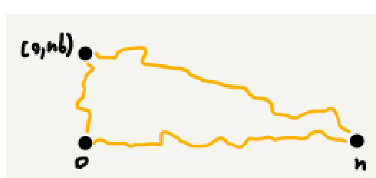
For every n construct the following measure $\widetilde{m}_n$ on $\widetilde{\Omega}$. Conditional on the edge weight distribution ω, it is a dirac measure on the associated geodesic and Busemann function. Find a subsequence n_k such that $\widetilde{m}_{n_k}$ converges to some $\widetilde{m}$. Do this in such a way that $\mathbb{Z}^2$ acts on $(\widetilde{\Omega}, \widetilde{m})$ in a measure preserving way.

By the nature of the lines L_n, this construction is already shift equivariant in the vertical direction. To make it shift equivariant horizontally, we will perform averaging over lines.



$$\widetilde{m}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{(\omega, \text{GEO}(v, L_i), \{B_i(v, v+e_1), B_i(v, v+e_2)\})}$$

Conditional on a configuration $\omega \in \Omega_1$, we get a dirac measure on $\Omega_2 \times \Omega_3$. For any such configuration, the space $\Omega_2 \times \Omega_3$ is compact. Pick a subsequence n_k such that $\widetilde{m}_{n_k} \to \widetilde{m}$.



Fix an ergodic component of $(\widetilde{\Omega}, \widetilde{m}, \{T_v\}_{v\in\mathbb{Z}^2}\}$. Consider

$$\mu_0 = \lim_{n\to\infty} \frac{1}{n} T((0,0),(n,0))$$
$$\mu_b = \lim_{n\to\infty} \frac{1}{n} T((0,bn),(n,0))$$

We claim that $\mathbb{E}B((0,0),(1,0)) = \mu_0$ and $\mathbb{E}B((0,0),(1,0)) = 0$. Once this claim is established, it will tell allow us to tell Busemann functions apart in every direction, and hence we can tell the geodesics apart as well.

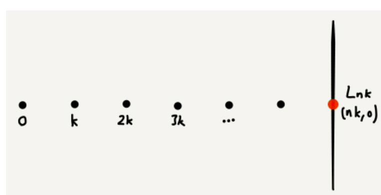
Recall that we are assuming that the shape is differentiable at the rightmost point: it is an exposed point of differentiability. Then the geodesic is going to be asymptotically in the horizontal direction. If instead there is a flat piece, then you are going to get a geodesic which is eventually contained in any slightly larger cone angle.

We want to show that $\mu_b = \mu_0(1 + o(b))$. What sorts of things will pass through the subsequential limit? Statements that hold with very high probability, like

$$\left|\frac{1}{k} T((0,0),(k,0)) - \mu_0\right| < \varepsilon.$$

From the shape theorem, we get

$$\mathbb{E}\left[\left(\frac{1}{k} B((0,0),(k,0)) - \mu_0\right) 1_{\text{bigger than } \varepsilon}\right] < \varepsilon'.$$



Here k is some fixed large number and consider the line L_{nk} where $n \to \infty$. Then $B_{nk}((0,0),(nk,0)) = T((0,0),L_{nk})$.

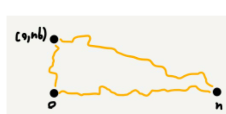
With high probability, for most i and n we have

$$B_{nk}((ik,0),((i+1)k,0)) = \mu_0 k(1 + o(1)).$$

Most means that for large enough k, I can make the fraction as close to 1 as desired.

Using a Fubini-type argument, we can show that for any fixed adjacent pair the Busemann function is close to $\mu_0 k$ for most lines. This says that when you take a subsequential limit, the expected value is μ_0 on any ergodic component.

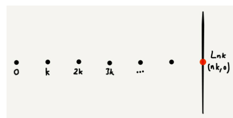
Let $c = \mathbb{E}B((0,0),(0,1))$. Consider the triangle with vertices $P_1 = (0,0)$, $P_2 = (n,0)$ and $P_3 = (bn,0)$.



Then the Busemann function satisfies

$$B(P_1, P_2) = \mu_0 n(1 + o(1)), \qquad B(P_1, P_3) = cn(1 + o(1)).$$

Choose b small and take n to ∞. Suppose for contradiction that $c < 0$.



Then

$$\begin{aligned} B(P_3, P_2) &= B(P_3, P_1) + B(P_1, P_2) \\ &= \mu_0 n(1 + o(1)) - cbn(1 + o(1)) \\ &= n(\mu_0 - cb)(1 + o(1)). \end{aligned}$$

But now $\mu_0 - cb > \mu_0$. But now we can bound by the passage time (that's all we know about Busemann functions: they are additive and bounded by the passage times).

$$T(P_3, P_1) = \mu_b n(1 + o(1)).$$

From the bound $T(P_3, P_2) \geq B(P_3, P_2)$, we get $\mu_0 - cb \leq \mu_b$. Saying that we have no sharp corner means that $\mu_b = \mu_0(1 + o(b))$. Combined with these inequalities yields the result.

FEBRUARY 19

A geodesic has a generalized direction a.s. $T(0,v) = B(0,v)$. If geodesic is $\{v_n\}$ then

$$\frac{T(0,v_n) - \mu_0 v_1}{|v_1| + |v_2|} \to 0.$$

Let $v_n = (v_{n,1}, v_{n,2})$. From the Busemann function,

$$T(0,v_n) = \mu_0 v_{n,1} + o(\|v_n\|_1).$$

From the shape theorem, $T(0,v_n) = \mu_0 v_{n,1} + o(\|v_n\|_1)$.

$$\mu_b = \lim_{n\to\infty} \frac{1}{n} T((0,0),(n,bn)) > \mu_0$$

since there is no flat piece. Here $b \approx v_{n,2}/v_{n,1}$.

Geodesics from different points are either disjoint or coalesce. Want to say that they coalesce with probability 1, when you look at geodesics to parallel lines. (This doesn't occur in the Euclidean metric, so we will need some assumptions on the edge weight model.)

With probability one, the geodesics are backwards finite. Let $B_0 = \{y\colon 0$ is on the geodesic from $y\}$. Then $P(|B_0| = \infty) = 0$. This is a true statement but not the one we want to prove.

Instead we want to show that $\mathbb{P}($0 is the least element of the equivalence class from the geodesic$) = 0$. In other words, if you go forward first, you can go back farther than where you started. The proof is a simple consequence of the mass transport principle.

The mass transport principle applies whenever you have a random function $f(x,y) \geq 0$ on $\mathbb{Z}^2$ which is shift invariant and satisfies $\sum_y \mathbb{E}f(x,y) < \infty$. The amount of mass that moves from x to y is $f(x,y)$. Then the mass out of x is $\sum_y \mathbb{E}f(x,y)$ which must equal the mass into x:

$$\sum_y \mathbb{E}f(x,y) = \sum_x \mathbb{E}f(x,z).$$

What function do we use? Put mass 1 at every point. Set

$$f(x,y) = \begin{cases} 1, & y \text{ is least element of equiv class of } x \\ 0, & else \end{cases}.$$

The expected amount sent to the least element is the sum over the mass sent out from the rest of the equivalence class, which must be 0 or ∞. By the mass transport principle it has to be finite, so it is 0. Hence there is no least element in the equivalence class.

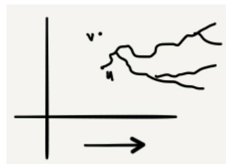
To prove the coalescence result, we need to ensure that there is enough randomness present. This is the updwards finite energy requirement.

For any edge e and a.e. configuration ω outside e, let $D_{\omega,e}$ be the distribution of $\omega(e)$ given ω on $E(\mathbb{Z}^2)$.

Then either there is no essential supremum, or if there is then it has 0 probability of equaling the supremum. For convenience, let's assume that it has unbounded support.

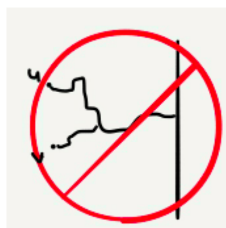
If they didn't coalesce, then there would be some interval $(0, 0)$ to $(0, n)$ containing three geodesics and none of them coalesce. In other words, some bounded interval intersects three equivalence classes. Then a topological argument shows that the trees can't cross and you get a contradiction of the minimal element.

February 22



Measure on tree of geodesics with direction positive x-axis non-crossing. Claim: we can construct a random geodesic with the following property: geoesics from any two points are either disjoint or coalescing. The tree of geodesic from v in direction of positive x axis is totally ordered by moving clockwise.

We will give a slightly different argument than the one written down by Damron and Hanson. Such a random geodesic divides $\mathbb{Z}^2$ into equivalence classes where $u \sim v$ if the geodesics starting from u and v eventually coalesce.



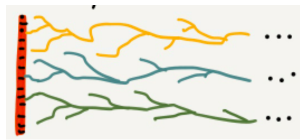
Suppose for contradiction that with positive probability, the geodesics from u and v do not coalesce. Equivalence classes have no leftmost element. We will make an equiv class with a leftmost element to get the contradiction.

The tree of geodesics along the positive x-axis is totally ordered by moving clockwise. Pick one of these geodesics in a stationary way: intuitively like picking the 'median'. These random geodesics divide $\mathbb{Z}^2$ into equivalence classes.

Here is the argument. First we suppose there are at least two equivalence classes. Then we boost this to get infinitely many equivalence classes. From here we show that there are at least 3 equivalence classes hitting some finite window. Now we apply the finite upward energy property:

For any set $A \subset \Omega$ and $E = \{e_1, \dots, e_n\} \subset E(\mathbb{Z}^2)$ and $\lambda > 0$. Define

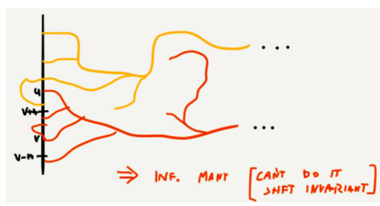
$$\widetilde{A} = \{\widetilde{\omega} \in \Omega \colon \exists \omega \in A, \widetilde{\omega}|_{E^c} = \omega|_{E^c}\}$$



Use the finite upward energy to build a wall on the y-axis. Make it more likely to go around. We didn't change anything for large x, so we still have an infinite geodesic in the same equivalence class.

We need another claim to argue that the end of the random geodesic is unaffected if we alter the configuration on finitely many edges. Thus we have succeeded in trapping the geodesic in between the wall and the two other geodesics.

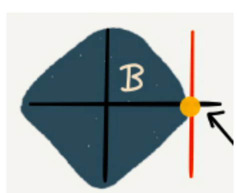
The density of the equivalence going off in any direction is 1.



Can't have one which is both of them. Have to have some highest thing in the infinite down. Has to be 0 by shift invariance, sum up over all so can't have an interface anywhere. Therefore the curve intersects the y-axis in a finite set.

What can we say about uniqueness?

Suppose the boundary of the shape has an exposed point of differentiability on the positive x-axis. Then there is a unique geodesic in the positive x-direction.



If there were two, then you can take the counterclockwise most one and the clockwise most one. By the properties of the limit shape's exposed point of differentiability, we know that they both point along the same direction. Then we have multiple coalescing geodesics with positive probability.

Consider the clockwise most and counterclockwise most. These are the random geodesics we talked about earlier. Damron-Hanson show that in this case, the Busemann function for one must have a different linear direction than that of the other.

February 24

Does there exist a model of fpp with a sharp corner in direction $\vec{v}$ where there is more than one gedesic in direction $\vec{v}$? Various teasers of the talks on further topics that will happen near the end of the quarter.

Let $t_n = T((0,0),(n,0))$. We know that $\mathrm{Var}(t_n)$ is between $\log n$ and $n/\log n$. The conjecture is that $\mathrm{Var}(t_n) = n^{\alpha+o(1)}$ for some $\alpha \in (0,1)$. A weaker form of this conjecture is that $\mathrm{Var}(t_n) < Cn^\alpha$ for some $\alpha < 1$. Now we will discuss how one might go about establish such a result.

Assumptions:

1. Edge weights are independent (probably not true without this assumption: can make it anything less than 2) (ergodic theorem gives an upper bound of 2)

2. Limiting shape has uniform curvature (e.g. spherically symmetric model)

Boundary of $B(0,t)$ should be almost Brownian in small pieces. But before we do that, let's look at the following picture.

Consider paths from $(0,0)$ to $(n,0)$. We want to give a criterion for when the point $(n/2,0)$ doesn't lie on the geodesic. Certainly if you can get to some $(n/2,k)$ to both the origin and $(n,0)$ than you can from $(n/2,0)$, then the geodesic will not go through $(n/2,0)$. Repeat this argument for $-k$ as well.

Take geodesics connecting these points. If v is outside of the four geodesics, then $(n/2,0)$ is not on the geodesic from $(0,0)$ to v. Prove this by contradiction: if you had such a path, then you can find a faster route along one of the legs. (This part of the argument uses the topology of $\mathbb{Z}^2$ crucially: because geodesics have codimension 1, which fails in higher dimensions.)

Enumerate the edges and set $\Sigma_k = \sigma(\omega_1,\ldots,\omega_k)$. We want to estimate

$$\mathbb{P}(e_{k+1} \text{ is on the geodesic} \mid \Sigma_k) < \ell^{-\beta}$$

for some $\beta > 0$ with probability at least $1 - n^{-\beta}$, where e_{k+1} has x coordinate ℓ.

Use an enumeration of all edges with x coordinates between 0 and n. Use a martingale decomposition of the variance. We won't be able to exactly get the picture from before, but we'll try to get something similar. The problem is that there will be a fastest path to the geodesic, so we won't get two geodesics. We relax the inequality so that they might pinch off at the end.

If the boundary was Brownian, when you move some distance j along the vertical midline then the fluctuation is on the order of $\sqrt{j}$. If there is an alpha, then it must be less than .99. So consider the farthest point away. We want to show that it is unlikely to get an edge there on the geodesic.

Consider a dyadic decomposition of edges

$$e, \quad e+(0,2), \quad e+(0,2^2), \quad \ldots, \quad e+(0,2^m)$$

and similarly on the negative side. It would be nice if they were at most a constant times the square root of the distance back. We could try to leverage this to get convergence to Brownian motion.

If they are not too far off, then on the right side they will be sufficiently faster than the path from the worst edge. Then we could apply a variation of the topological argument from before to conclude that the bad edge couldn't lie on the geodesic out to any far away v (in particular $(0, n)$).

Now $m < c \log n$ because for $2^m < \sqrt{n}$ the Euclidean distance doesn't change more than a constant between the leg and the hypotenuse.

Why exponentially spaced out? We need two things to happen. First we need a reasonable (bounded away from 0) chance that they pinch off at $(n, 0)$. We need them to be pretty independent of one another. Going out an exponential distance achieves this.

Now we want to consider the number of good pairs of lower and upper points that will have a reasonable chance of cutting off the geodesic. Then the probability that e is on the boundary is less than $p^{c \log n}$ for some $p < 1$.

FEBRUARY 26

1. independent edge weights
2. limiting shape is a disk

Goal: Prove that $\mathrm{Var}(T((0,0),(n,0))) \approx n^\beta$ for $\beta < 1$.

Reveal the passage times to the left of the line $x = j$ for $j > 0$. Fix $(0,j)$ and consider $v + (0, 2^k)$ and $v - (0, 2^k)$. Assume that $\partial B(t, 0)$ is roughly Brownian on scales up to k.

If every point on the line $x = j$ satisfies $(\star)$

$$c \log \inf(j, n-j) \text{ choices of } k \in \left\{1, \dots, \frac{1}{2}\log j\right\}$$

such that $\{y = v \pm (0, 2^k)\} \cap \partial B(0, T(0,v))$ is no more than $100 \cdot 2^{k/2}$ behind $\{y = v\} \cap \partial B(0, T(0,v))$. (Note that we will really need $\inf(j, n-j)$ everywhere but we will stop writing it now.)

If the boundary has square root fluctuations then $(\star)$ should hold with high probability for every vertex.

Suppose now that $(\star)$ holds for some vertex v on $x = j$. Then

$$\mathbb{P}(v \in \textsc{geo}((0,0),(n,0))) < j^{-\alpha}, \qquad \alpha > 0.$$

To see why, let γ be the probability that they pinch off the geodesic (i.e., the passage times are faster). Since these are roughly independent, the probability that roughly none of them happen is

$$\gamma^{c \log j} = j^{c \log \gamma} = j^{-\alpha}.$$

Note that γ would be independent of k if the boundary was Brownian, by scaling.

Now we will talk about how to try to run this argument without knowing that the boundary has Brownian fluctions. Assume that $\mathrm{Var}(T(0,0),(n,0)) \approx n^\beta$ for some $\beta < 1$.

Once the geodesics split, they will be essentially independent. There will be a sweet spot which balances the variance against the distance savings. Consider v and $v + (0,m)$ for $m < \sqrt{|v|}$. Let ℓ denote the distance where they split off. Then the fluctuation is $\ell^{\beta/2}$, which should be on the same order as $\sqrt{\ell^2 + m^2} - \ell$, which is on the order of m^2/ℓ.

Solving gives $\ell \approx m^{4/(2+\beta)}$. Since we expect $\beta = 2/3$, we expect $\ell \approx m^{3/2}$. The fluctuations are on the order $\ell^{\beta/2} \approx m^{2\beta/(2+\beta)}$ which is $\sqrt{m}$ when $\beta = 2/3$.

Given m, define $\ell(m)$ by the equation

$$\sqrt{\ell^2 + m^2} - \ell(m) \approx \sqrt{\mathrm{Var}\, T(0,(m,0))}.$$

Note that since we're in the spherically symmetric case, we have a bound on β to the effect of $\beta \in (1/4, 1)$. Thus the approximation $\sqrt{\ell^2 + m^2} - \ell(m) \approx m^2/2\ell$ is justified.

In terms of our earlier k, this quantity is $2^k/2\ell(2^k)$. Now we run the earlier argument in terms of these more general quantities. Put a box around $v+(0,2^k)$ of height $.1 \cdot 2^k$ and width $\ell(2^k)$. Since the boxes are disjoint, what happens inside is independent. With probability bounded uniformly below (using scale invariance), there is a path in the box that goes really fast: there is a geodesic in the box that goes 1000 standard deviations faster than the mean.